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Discrete symmetries of differential equations

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Abstract. The determination of the continuous symmetries of differential equations follows a well known algorithm, and is reduced to solution of a set of linear equations; this is based on considering infinitesimal generators of the symmetries, so that the method does not extend to discrete symmetries. In this paper, we present a method to determine discrete symmetries in a certain class by means of a linear system, although this is considerably more difficult to solve than the one connected with continuous symmetries. We also consider the inverse (and simpler) problem of determining the most general equation admitting a given discrete symmetry. In the last part, we consider a number of examples, dealing in particular with symmetries of relevance to physics.

1. Introduction

It is well known [1–7] that knowledge of continuous symmetries of differential equations (DEs) can be of considerable value in finding particular or general solutions of the equations, or in simplifying them.

In the determination of *continuous* symmetries of DEs, we proceed by identifying the DE Δ with a manifold S_Δ in the appropriate *jet space* $\mathcal{J}$; we then look for a vector field η_0 such that its prolongation η is tangent to S_Δ. In this way we are reduced to considerations on the *tangent space* to S_Δ, i.e. to a *linear* problem. Indeed, the condition $\eta : S_\Delta \to TS_\Delta$ gives a system of linear PDEs, the *determining equations* [1–7].

Knowledge of *discrete* symmetries would also be of great use in the study of DEs; unfortunately, in the determination of general discrete symmetries we cannot reduce to study the infinitesimal action of vector fields, and we end up in general with a nonlinear problem.

In this paper, we point out that for some class of discrete symmetries, which we will call *quantized* or *stroboscopic* for reasons which will become clear in what follows, we can still reduce to a linear problem, although considerably more difficult than the one to be solved for continuous symmetries. The method we propose has a clear geometric interpretation, and indeed we will present it geometrically, starting from the identification of Δ with the solution manifold S_Δ.

The main ideas of our approach have been presented in a previous short note [8]; here we give a more detailed discussion and expand in several directions; we also consider explicit examples and special cases of physical relevance.

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The class of discrete transformations we will study is that of transformations obtained by finite action of vector fields; i.e. if η_0 is a vector field, we will consider transformations of the form $T_\lambda = e^{\lambda\eta}$, where $\lambda \in R$ is a *finite* parameter; we are interested in the case η is *not* a symmetry of Δ, but there exist values λ_0 such that $T_{\lambda_0} : S_\Delta \to S_\Delta$ (and $T_{\lambda_0} \neq I$). An example of this would be an equation which is invariant under a lattice of discrete translation, but not under continuous translations, see below.

It should be anticipated that the determining equation we obtain by the method proposed here is a *functional* equation, so that in general we cannot hope to find the most general solution, i.e. the complete set of discrete symmetries to a given DE. It should also be remarked that the problem is unavoidably underdetermined: indeed, two different vector fields η and η' such that $e^{\lambda_0\eta} = e^{\lambda_0\eta'}$ give raise to the same discrete transformation; we will comment on this further in what follows.

It should also be mentioned that the problem of finding continuous symmetries for discrete equations, which is in some sense dual to the present one (see below), has been considered by several authors [9–13]; in any case our method is different from those proposed to that end, and we will discuss the possibility (and limitations) of application of our method to such a problem.

We will suppose that the reader is familiar with the problem and language of determining continuous symmetries of differential equations, and in general with the symmetry theory of differential equations [1–7].

Let us briefly describe the plan of the paper. We will first consider the geometric problem of quantized transformations that leave a manifold in R^{n+1} invariant, and we will write the determining equations for these explicitly. We will then specialize to the case of the manifold in a jet space corresponding to a DE: in this case some extra structure is present, and this results in some simplification of the determining equation. Even with this, we are not able to give the complete solution to the determining equations, but we will show that by restricting the form of the vector field to be quantized we can give explicit solutions. We will then consider the inverse problem: that is, given a quantized symmetry, determine the most general DE (for assigned order and number of variables) that is invariant under that symmetry. This problem leads to a system of PDEs and can be completely solved in a number of cases, as we will discuss. We will then discuss the relation of our problem with the dual situation, recalled above, of continuous symmetries and discrete equations. At this point, we will discuss the geometrical interpretation of our method from a rather abstract point of view; this could be helpful in establishing relations with differential geometric properties of the manifold considered. Indeed, in this language we have to determine connections for a certain fibre bundle. Finally, we will consider in detail some explicit examples, focusing in particular on simple discrete symmetries of wide use, such as translations, rotations and scale transformations.

2. Quantized symmetries for manifolds

Let us consider $X = R^n$, $Y = R^1$. Let us consider a manifold Γ in $M = X \times Y = R^{n+1}$ defined by the equation

$$y = f(x) \tag{2.1}$$

with $f : X \to Y$ a smooth function.

Let us further consider a vector field in M

$$\eta = \varphi^i(x, y)\partial_i + \psi(x, y)\partial_y \,. \tag{2.2}$$

For brevity here and in what follows we write

$$\partial_i \equiv \frac{\partial}{\partial x^i} \tag{2.3}$$

and summation over repeated indices is understood.

The infinitesimal action $e^{\varepsilon\eta}$ of η in M maps (x, y) to a new point (x', y'), where

$$x' = x + \varepsilon\varphi(x, y) \qquad y' = y + \varepsilon\psi(x, y) \tag{2.4}$$

so that the graph of f, $\Gamma = \{(x, f(x))\}$ is transformed into a new curve $\Gamma_\varepsilon = \{(x, f_\varepsilon(x))\}$ which is the graph of the function

$$f_\varepsilon(x) = f(x) + \varepsilon[\psi(x, y) - (\varphi^i(x, y)\ \partial_i) f(x)]\,. \tag{2.5}$$

Let us now introduce the function $F : R \times X \to R$ such that $F(\lambda, x) = f_\lambda(x)$ is the transform of $f(x)$ under $e^{\lambda\eta}$. Clearly, the infinitesimal transformation (2.5) yields that this F satisfies the *determining equation*

$$\frac{\partial F(\lambda, x)}{\partial\lambda} + \varphi^i(x, F(\lambda, x))\ \frac{\partial F(\lambda, x)}{\partial x^i} = \psi(x, F(\lambda, x)) \tag{2.6}$$

with initial condition

$$F(0, x) = f(x)\,. \tag{2.7}$$

If η is a continuous symmetry of f, then Γ must be invariant under η, i.e. $f_\varepsilon(x) = f(x)$, or equivalently $(\partial F/\partial\lambda) = 0$. Indeed in this case (2.6) gives just the usual determining equations for the symmetries of (2.1).

As mentioned in the introduction, we are interested in the case where η is not a symmetry of (2.1), but there is a special value λ_0 such that $f_{\lambda_0} = f$, i.e. such that

$$F(\lambda_0, x) = F(0, x)\,. \tag{2.8}$$

In other words, we are interested in determining η such that the solution to (2.6) with the initial condition (2.7) is *periodic* in λ (excluding the trivial case $\partial F/\partial\lambda = 0$). In this case, $\Lambda = e^{\lambda\eta} : M \to M$ maps Γ into itself and therefore qualifies as a discrete symmetry of (1), or equivalently of Γ.

It should be stressed that we also must require that $\Lambda|_\Gamma$ is not the identity, or we would have a trivial discrete symmetry.

Remark 1. It is now clear why we call such a symmetry, given by finite action of a vector field which is not a continuous symmetry of (2.1) itself, a quantized or stroboscopic symmetry, as anticipated in the introduction.

Remark 2. By multiplying η by a numerical constant, we can always set $\lambda_0 = 1$, or $\lambda_0 = 2\pi$. Thus, in what follows we will in general consider λ_0 as fixed. The (2π, or any fixed period) periodicity requirement in λ would suggest to expand $F(\lambda, x)$ as a Fourier series in λ. This is indeed possible, but is of practical use only if (2.6) is linear (see below).

Remark 3. We stress that in (2.6) we have to determine not only F, but φ and ψ as well. Thus, we are not dealing with a normal PDE, but with a functional PDE. The only data are the initial condition (2.7) and the (arbitrary) $\lambda_0 \neq 0$ appearing in (2.8).

Remark 4. We could—and will—consider the inverse problem of determining the manifolds Γ which are invariant under the action of a given prescribed symmetry. In this case φ, ψ are given, and (6) is a regular PDE, not a functional one. It is not surprising that this inverse problem is much easier than the direct one (see below).

Remark 5. Although we have considered quite special X, Y, M and Γ for ease of notation, it is simple to generalize the above discussion to the case where X and Y are smooth submanifolds in real spaces; for our purposes we will not need to consider the case Γ is a generic smooth submanifold in $M = X \times Y$, and only need to consider Γ as the graph of a function f in an appropriate space.

3. Quantized symmetries of differential equations

As already recalled, a DE Δ is naturally identified with a manifold S_Δ in an appropriate jet space $\mathcal{J}$ [1, 6, 7, 15, 17]. Therefore, if Γ is S_Δ and M is $\mathcal{J}$, the theory developed in the previous section can be applied to differential equations as well.

However, the fact that we are dealing with jet spaces makes that an extra structure (the contact structure) is now present. Due to this, we have some extra constraints on the functions appearing in (2.6): e.g., η should now be the prolongation to $\mathcal{J}$ of an underlying Lie-point vector field η_0 acting in the space M_0 of independent and dependent variables; due to this the φ^i and ψ are not arbitrary smooth functions, but must satisfy some relations (embodied in the prolongation formula [1–7]), as we are now going to discuss.

3.1. Ordinary differential equations

Let us first consider the case of an autonomous ODE

$$u_t = f(u) . \tag{3.1}$$

and time-independent Lie-point vector field

$$\eta_0 = \varphi(u)\partial_u . \tag{3.2}$$

Now $M_0 = R^2 = \{(t, u)\}$, and (3.1) is identified with a manifold in $\mathcal{J} = M = R^3 = \{t, u, u_t\}$. The prolongation of $\eta_0 : M_0 \to TM_0$ to M is given by

$$\eta = \varphi(u)\partial_u + \Phi(u, u_t)\partial_{u_t} \tag{3.3$'$}$$

where $\Phi = \varphi_u u_t$ by the prolongation formula [1–7], so that on Γ we have

$$\Phi = \varphi_u f(u) . \tag{3.3$''$}$$

Now u_t plays the role of y in section 2, and Φ corresponds to ψ; thus we get equation (2.6) in the form

$$\frac{\partial F(\lambda, u)}{\partial \lambda} + \varphi(u)\left(\frac{\partial F(\lambda, u)}{\partial u}\right) = \psi = \varphi_u F(\lambda, u) \tag{3.4}$$

(we have taken t-independence into account). Thus, ψ is now determined by φ, and we have only one arbitrary function in η. Note also that while the φ in the case of section 2 could depend on y, now the request that η_0 be a Lie-point vector field ensures that φ does not depend on u_t, and we end up with a determining equation which is linear in F.

Thus, dealing with differential equations rather than algebraic manifolds does indeed give a somewhat simpler problem!

It also is worth considering the case of non-autonomous ODEs and time-dependent vector fields, i.e.

$$u_t = f(t,u) \tag{3.5}$$

$$\eta_0 = \tau(t,u)\partial_t + \varphi(t,u)\partial_u\,. \tag{3.6}$$

In this case η is still given by (3.3′), but the prolongation formula yields $\Phi = \varphi_t + \varphi_u u_t - \tau_t u_t - \tau_u (u_t)^2$, and on Γ we have

$$\Phi = \varphi_t + [\varphi_u - \tau_t] f(t,u) - \tau_u [f(t,u)]^2 \tag{3.6′}$$

so that we end up with the determining equation

$$\frac{\partial F}{\partial \lambda} + \tau(t,u)\frac{\partial F}{\partial t} + \varphi(t,u)\frac{\partial F}{\partial u} = \psi = \varphi_t + [\varphi_u - \tau_t]F - \tau_u F^2\,. \tag{3.7}$$

The appearance on the right-hand side of a nonlinear term is peculiar to the case of first-order ODEs: indeed, by the recursive structure of the general prolongation formula [1–7] one can easily see that for general ODEs of order n the determining equations will have on the left-hand side a differential operator whose coefficients do not depend on F (while in the case of algebraic manifolds, see section 2, they could depend on F), applied on F, and on the right-hand side an expression which contains terms of order not higher than one in F if $n \neq 1$, and not higher than two in F if $n = 1$.

Remark 6. The fact that first-order differential equations lead to more difficult determining equations than higher-order ones should not be a surprise. Indeed, the same happens also in the case of determining equations for continuous Lie-point symmetries [1–7].

The general procedure for writing the determining equations for discrete symmetries of higher-order ODEs or evolution PDEs should be clear by now; it amounts to repeating the procedure illustrated in section 2 for manifolds and taking into account that $\eta : M \to TM$ is now the prolongation of $\eta_0 : M_0 \to TM_0$. We stress that this only requires to apply the general prolongation formula [1–7].

We will give the formulae for the case of second-order ODEs and of evolution PDEs of order one or two in the spatial derivatives explicitly.

In the case of general (non-autonomous) second-order ODE

$$u_{tt} = f(t,u,u_t) \tag{3.8}$$

and general η_0, the η_0 is as in (3.6), and the second prolongation of this yields

$$\begin{aligned}\eta = {} & \tau(t,u)\partial_t + \varphi(t,u)\partial_u + [\varphi_t + (\varphi_u - \tau_t)u_t - \tau_u(u_t)^2]\partial_{u_t} \\ & + [\varphi_{tt} + (\varphi_u - 2\tau_t - 3\tau_u u_t)u_{tt} + (2\varphi_{ut} - \tau_{tt})u_t \\ & + (\varphi_{uu} - 2\tau_{ut})(u_t)^2 - \tau_{uu}(u_t)^3]\partial_{u_{tt}}\end{aligned} \tag{3.9}$$

so that the determining equation is

$$\frac{\partial F}{\partial \lambda} + \tau \frac{\partial F}{\partial t} + \varphi \frac{\partial F}{\partial u} + [\varphi_t + (\varphi_u - \tau_t) u_t - \tau_u (u_t)^2] \frac{\partial F}{\partial u_t} = \varphi_{tt} + (\varphi_u - 2\tau_t - 3\tau_u u_t) u_{tt} + (2\varphi_{ut} - \tau_{tt}) u_t + (\varphi_{uu} - 2\tau_{ut})(u_t)^2 - \tau_{uu}(u_t)^3 . \tag{3.10}$$

For the special case of an autonomous second-order ODE

$$u_{tt} = f(u, u_t) \tag{3.11}$$

and time independent η_0 [as in (3.2)], we get

$$\eta = \varphi \partial_u + \varphi_u u_t \partial_{u_t} + [\varphi_{uu}(u_t)^2 + \varphi_u u_{tt}] \partial_{u_{tt}} \tag{3.12}$$

and the determining equation is therefore

$$\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + \frac{\partial \varphi}{\partial u} \frac{\partial F}{\partial u_t} (u_t) = \frac{\partial^2 \varphi}{\partial u^2} (u_t)^2 + \frac{\partial \varphi}{\partial u} F \tag{3.13}$$

where $\varphi = \varphi(u)$ and $F = F(\lambda, u, u_t)$.

3.2. Partial differential equations

For a first-order PDE

$$u_t = f(t, x, u, u_x) \tag{3.14}$$

with $x \in R^q$, we write

$$\eta_0 = \tau(t, x, u) \partial_t + \xi^j(t, x, u) \partial_j + \varphi(t, x, u) \partial_u \tag{3.15}$$

where j runs from 1 to q, summation over repeated indices is understood, and $\partial_j = \partial / \partial x^j$. The first prolongation of this yields

$$\eta = \tau(t, x, u) \partial_t + \xi^j(t, x, u) \partial_j + \varphi(t, x, u) \partial_u + [\varphi_t + \varphi_u u_t - \tau_t u_t - \tau_u (u_t)^2 - \xi^j_t u_j - \xi^j_u u_j u_t] \partial_{u_t} + [\varphi_j + \varphi_u u_j - \tau_j u_t - \tau_u u_t u_j - \xi^i_j u_i - \xi^i_u u_i u_j] \partial_{u_j} \tag{3.16}$$

and therefore the determining equation is (note that F now also appears in the coefficients of F derivatives on the left-hand side of the determining equation, in contrast to what happens in the case of ODEs):

$$\frac{\partial F}{\partial \lambda} + \tau \frac{\partial F}{\partial t} + \xi^j \frac{\partial F}{\partial x^j} + \varphi \frac{\partial F}{\partial u} + \left[\varphi_j + \varphi_u u_j - (\tau_j + \tau_u u_j) F - \xi^i_j u_i - \xi^i_u u_i u_j \right] \left(\frac{\partial F}{\partial u_j} \right) = \left[\varphi_t - \xi^j_t u_j + (\varphi_u - \tau_t - \xi^j_u u_j) F - \tau_u F^2 \right] . \tag{3.17}$$

For autonomous first-order PDEs

$$u_t = f(u, u_x) \tag{3.18}$$

and for η_0 independent of x and t,

$$\eta_0 = \varphi(u) \partial_u \tag{3.19}$$

we get

$$\eta = \varphi(u) \partial_u + (\varphi_u u_x) \partial_{u_x} + (\varphi_u u_t) \partial_{u_t} \tag{3.19$'$}$$

and the determining equation is

$$\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + (\varphi_u u_x) \frac{\partial F}{\partial u_x} = \varphi_u F . \tag{3.20}$$

We stress that in this case the coefficient of F derivatives on the left-hand side of the determining equation do not depend on F, and the determining equation is *linear*.

Remark 7. By looking at the prolongation formula, we see at once that this is a general fact; i.e. for autonomous PDEs and $\eta_0 = \varphi(t, x, u)\partial_u$, the determining equations are always linear.

Remark 8. If we have an autonomous evolution equation, the general form of η_0 transforming it into autonomous evolution equations is just $\eta_0 = \tau(t)\partial_t + \xi^j(x)\partial_j + \varphi(t, x, u)\partial_u$ [6]. Note that for autonomous equations, the time and space translations correspond to continuous symmetries, and so are not of interest in the present context.

In the case of second-order evolution PDEs, for ease of notation we will keep to autonomous equations

$$u_t = f(u, u_x, u_{xx}) \tag{3.21}$$

and η_0 as in (3.19). In this case we get

$$\eta = \varphi\partial_u + \varphi_u u_x \partial_{u_x} + (\varphi_{uu}(u_x)^2 + \varphi_u u_{xx})\partial_{u_{xx}} + \varphi_u u_t \partial_{u_t} \tag{3.22}$$

and the determining equation is therefore

$$\frac{\partial F}{\partial \lambda} + \varphi\ \frac{\partial F}{\partial u} + (\varphi_u u_x)\frac{\partial F}{\partial u_x} + \left[\varphi_{uu}(u_x)^2 + \varphi_u u_{xx}\right]\frac{\partial F}{\partial u_{xx}} = \varphi_u F\,. \tag{3.23}$$

3.3. *Quantized symmetries with linear generators*

A special case of interest is the one where the functions appearing in the vector field η_0 are *linear*. We are going to discuss this case now, limiting ourselves to evolutionary symmetries [1–7], i.e. to the case

$$\eta_0 = \varphi_\alpha(u)\frac{\partial}{\partial u_\alpha} \tag{3.24}$$

where linearity means

$$\varphi_\alpha(u) = A_{\alpha\beta}u_\beta\,. \tag{3.25}$$

From the previous computations we get that for first-order ODEs (autonomous or not)

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}t} = f_\alpha(u, t) \tag{3.26}$$

the determining equations in the present case are

$$\frac{\partial F_\alpha}{\partial \lambda} = F_\beta\frac{\partial \varphi_\alpha}{\partial u_\beta} - \varphi_\beta\frac{\partial F_\alpha}{\partial u_\beta}\,. \tag{3.27}$$

For second-order ODEs

$$\frac{\mathrm{d}^2u_\alpha}{\mathrm{d}t^2} = f_\alpha(u, u_t; t) \tag{3.28}$$

the determining equations have a similar form (here $u^j \equiv u_j$, $u_t^j = \mathrm{d}u_j/\mathrm{d}t$):

$$\frac{\partial F_\alpha}{\partial \lambda} + A_{\gamma\beta} u_t^\beta \frac{\partial F_\alpha}{\partial u_t^\gamma} = F_\beta \frac{\partial \varphi_\alpha}{\partial u_\beta} - \varphi_\beta \frac{\partial F_\alpha}{\partial u_\beta} . \tag{3.29}$$

Let us now consider first-order PDEs:

$$\frac{\partial u_\alpha}{\partial t} = f_\alpha(u, u_x; x, t) \tag{3.30}$$

and denote $u_x^j \equiv \partial u_j/\partial x$. These yield the determining equations

$$\frac{\partial F_\alpha}{\partial \lambda} + A_{\gamma\beta} \frac{\partial u_\beta}{\partial x} \frac{\partial F_\alpha}{\partial u_x^\gamma} = F_\beta \frac{\partial \varphi_\alpha}{\partial u_\beta} - \varphi_\beta \frac{\partial F_\alpha}{\partial u_\beta} . \tag{3.31}$$

Finally, for (autonomous) second-order evolution PDEs

$$\frac{\partial u_\alpha}{\partial t} = f_\alpha(u, u_x, u_{xx}) \tag{3.32}$$

the determining equations are

$$\frac{\partial F_\alpha}{\partial \lambda} + A_{\gamma\beta} u_x^\beta \frac{\partial F_\alpha}{\partial u_x^\gamma} + A_{\gamma\beta} u_{xx}^\beta \frac{\partial F_\alpha}{\partial u_{xx}^\gamma} = F_\beta \frac{\partial \varphi_\alpha}{\partial u_\beta} - \varphi_\beta \frac{\partial F_\alpha}{\partial u_\beta} . \tag{3.33}$$

4. Differential equations with prescribed quantized symmetry

Determining the discrete (quantized) symmetries of a given differential equation along the lines of section 3, i.e. solving the determining equations obtained there, is a very difficult problem, as it amounts to solving a *functional* equation: indeed, we have to determine *both* the symmetry vector field η_0 (i.e. the functions φ, ξ, τ), *and* the function $F(\lambda, u)$ satisfying $F(0, u) = f(u)$.

The situation is substantially simpler if we prescribe a quantized symmetry *a priori*, and we try to determine the differential equations which admit this quantized symmetry.

Indeed, now φ is given, so that the determining equations become standard PDEs for $F(\lambda, u)$, and once $F(\lambda, u)$ has been determined, we just have to put $\lambda = 0$ (or actually $\lambda = \lambda_0$ for any constant λ_0) to have the most general $f(u)$ identifying a differential equation admitting the required quantized symmetry.

Also, looking at the determining equations written in section 3 for different classes of problems, we note that for autonomous equations, the determining equations are a system of quasilinear PDEs for $F(\lambda, u)$.

It should also be mentioned that the class of vector fields we have been considering so far is too ample from the physical point of view. Indeed, if we allow general vector fields of the form

$$\eta = \varphi(u, x, t)\partial_u + \xi(u, x, t)\partial_x + \tau(u, x, t)\partial_t \tag{4.1}$$

these will in general transform evolution equations into equations which are not of evolutionary type. In order to preserve the evolutionary character of the equations under the action of the Lie group, we should require [6] that this be 'fibre preserving', i.e. that the fibred structure of the space $M = X \times U$ be preserved. This is obtained by requiring [6] that η be of the form

$$\eta = \varphi(u, x, t)\partial_u + \xi(x, t)\partial_x + \tau(x, t)\partial_t \,. \tag{4.2}$$

Note that if we also require that the reparametrization of time should not depend on the space point (another physically reasonable assumption, out of relativistic theories), we should have a more restrictive class of vector fields, namely

$$\eta = \varphi(u, x, t)\partial_u + \xi(x, t)\partial_x + \tau(t)\partial_t \,. \tag{4.3}$$

However, whenever $\tau_u = 0$ (as it is the case for (4.2) or (4.3) above), the determining equations become *linear* in F, even when considering non-autonomous evolution problems.

We will now restrict consideration to the class of vector fields (4.3). The following table, associating with any of the class of equations considered there the determining equations as equations for F, can then be derived immediately; examples of solutions to these for vector fields η_0 of physical relevance will be given in section 9.

Evolution equation	Determining equation
$u_t = f(u)$	$F_\lambda + \varphi F_u = [\varphi_u]F$
$u_t = f(u; t)$	$F_\lambda + \tau F_t + \varphi F_u = \varphi_t + [(\varphi_u - \tau_t)]F$
$u_{tt} = f(u, u_t)$	$F_\lambda + \varphi F_u + (\varphi_u u_t)F_{u_t} = [\varphi_u]F + \varphi_{uu} u_t^2$
$u_{tt} = f(u, u_t; t)$	$F_\lambda + \tau F_t + \varphi F_u + (\varphi_t + (\varphi_u - \tau_t)u_t)F_{u_t}$ $= \varphi_{tt} + (\varphi_u - 2\tau_t - 3\tau_u u_t)u_{tt} + (2\varphi_{ut} - \tau_{tt})u_t$ $+ (\varphi_{uu} - 2\tau_{ut})(u_t)^2 - \tau_{uu}(u_t)^3$
$u_t = f(u, u_x)$	$F_\lambda + \varphi F_u + (\varphi_u u_x)F_{u_x} = \varphi_u \; F$
$u_t = f(u, u_x; t, x)$	$F_\lambda + \xi F_x + \varphi F_u + (\varphi_x + \varphi_u u_x - \xi_x u_x)F_{u_x}$ $= [\varphi_u - \tau_t - \xi_x u_x] \; F + \varphi_t - \xi_t u_x$
$u_t = f(u, u_x, u_{xx})$	$F_\lambda + \varphi F_u + (\varphi_u u_x)F_{u_x} + (\varphi_{uu} u_x^2 + \varphi_u u_{xx})F_{u_{xx}} = \varphi_u \; F$

5. Fourier expansion and the determining equation

Let us come back to the determining equation (2.6); as remarked in section 2, the periodicity condition (2.8) suggests to use a Fourier expansion. Here we will indeed look for solutions of this equation in such a form (we will choose $\lambda_0 = 2\pi$ for ease of notation), i.e.

$$F(\lambda, x) = \sum_{k=-\infty}^{\infty} c_k(x) \mathrm{e}^{\mathrm{i}k\lambda} \tag{5.1}$$

(note that if we are dealing with real F we should require $c_{-k} = c_k^*$). Substituting this in the determining equation we get

$$\sum_{k=-\infty}^{\infty} \left(\mathrm{i}k c_k(x) + \varphi^j(x, F(\lambda, x)) \frac{\partial c_k(x)}{\partial x^j} \right) \mathrm{e}^{\mathrm{i}k\lambda} = \psi(x, F(\lambda, x)) \,. \tag{5.2}$$

Setting the determining equation in this form is particularly useful if φ does not depend on F, and ψ contains only linear terms in F (and maybe terms independent of F as well). Indeed, in this case we get a series of decoupled equations for the Fourier coefficients $c_k(x)$.

In the case of general manifolds in R^n we have no reason to suppose this simplifying conditions are satisfied, but in the case of differential equations, as discussed in the previous section 3, we are guaranteed to be in such a situation (this is indeed why we deferred the discussion of Fourier approach up to now). We will therefore assume $\varphi^j = \varphi^j(x)$, and write ψ as

$$\psi(x, F(\lambda, x)) = \psi_0(x) + \psi_1(x)F(\lambda, x)\,. \tag{5.3}$$

Equating the coefficients of the functions $e^{i\lambda k}$ in both sides of the determining equations, we get the PDEs system

$$\varphi^j(x)\ \frac{\partial c_k(x)}{\partial x^j} + ikc_k(x) = \psi_0(x)\delta_{k0} + \psi_1(x)c_k(x) \tag{5.4}$$

which can equivalently be expressed as the eigenvalue problem

$$\left(\varphi^j(x)\frac{\partial}{\partial x^j} - \psi_1(x)\right)c_k(x) = -ikc_k(x) \tag{5.5}$$

for any $k \neq 0$. When $k = 0$ we get the inhomogeneous equation

$$\varphi^j(x)\frac{\partial c_0(x)}{\partial x^j} - \psi_1(x)c_0(x) = \psi_0(x)\,. \tag{5.6}$$

The periodicity condition (2.8) is now automatically satisfied; as for the initial condition (2.7), this now reads

$$F(0, x) = \sum_{k=-\infty}^{\infty} c_k(x) = f(x)\,. \tag{5.7}$$

The equations for the coefficients $c_k(x)$ are now, as already mentioned, linear and uncoupled.

Remark 9. Since we deal with a linear equation, its solutions will be given by an arbitrary superposition of a particular solution of the equation and any solution of the associated homogeneous equation (obtained by setting $\psi_0 = 0$). This shows that in this case we have a great degree of arbitrarity in solving the determining equation; this is indeed reflected in the fact that we can choose any combination of the c_k to give the F.

Remark 10. We should require $c_1(x) \neq 0$, or the periodicity of F would not be the required one (we recall $\lambda_0 \neq 0$ could be chosen arbitrarily by a rescaling). This suggests that, in order to reduce the arbitrariness in the choice of F somewhat, we could fix $c_k = 0$ for $|k| > 1$. This prescription leads to particularly simple computations, but not to the most general solution. Such a prescription amounts to writing $F(\lambda, x) = F_0(x) + F_1(x)\cos(\lambda) + F_2(x)\sin(\lambda)$ and in this form can also be implemented directly on (2.6); the initial condition (2.7) is now simply $F_1(x) + F_2(x) = f(x)$.

Remark 11. If we have $\partial F/\partial\lambda = 0$ (so that η_0 is a continuous symmetry), we get precisely the equations (5.4) without the $\mathrm{i}kc_k$ term. In this case, for $k \neq 0$ we get (5.5) with zero right-hand side, while for $k = 0$ we still get (5.6). The c_0 solution of (5.6) would actually give equations with continuous invariance, which are not of interest here. We can therefore concentrate on the $k \neq 0$ equations alone. With the above prescription, this means that we can focus on the $k = 1$ equation alone.

Let us now consider the 'inverse problem' mentioned above, i.e. let us consider the symmetry as given (that is, we know the functions $\varphi^j(x)$ and ψ). Each equation (5.5) is a linear first-order PDE and can be solved using the characteristics

$$\frac{\mathrm{d}x^1}{\varphi^1} = \frac{\mathrm{d}x^2}{\varphi^2} = \cdots = \frac{\mathrm{d}x^n}{\varphi^n} = \frac{\mathrm{d}c_k}{(\psi_1 - \mathrm{i}k)c_k} \,. \tag{5.8}$$

Solving the first set of equations,

$$\frac{\mathrm{d}x^1}{\varphi^1} = \frac{\mathrm{d}x^2}{\varphi^2} = \cdots = \frac{\mathrm{d}x^n}{\varphi^n} \tag{5.9}$$

we get a set of invariants, which are the invariants for the continuous symmetry η:

$$g_i(x) = 0 \qquad i = 1, \ldots, n \,. \tag{5.10}$$

We are then left with one further equation (here m is any integer between 1 and n, which we can choose as we please, provided $\varphi^m \neq 0$),

$$\frac{\mathrm{d}x^m}{\varphi^m} = \frac{\mathrm{d}c_k}{(\psi_1 - \mathrm{i}k)c_k} \tag{5.11}$$

which yields

$$c_k(x) = A_k(g_1(x), \ldots, g_n(x)) \exp\left[\int \left[(\psi_1(x) - \mathrm{i}k)/\varphi^m(x)\right] \mathrm{d}x^m\right] \tag{5.12}$$

where the A_k are arbitrary functions of the invariants (5.10).

The solution to the problem of determining DEs with prescribed quantized symmetries is therefore provided in this case by (5.1) with the $c_k(x)$ given by (5.12), m is such that $\varphi^m \neq 0$ and the A_k in (5.12) are such that (5.7) is satisfied.

6. Discrete symmetries of dynamical systems

We would like to consider in detail the discrete symmetries of dynamical systems, i.e. of *systems* of first-order autonomous equations,

$$\frac{\mathrm{d}u^i}{\mathrm{d}t} = f^i(u^1, \ldots, u^n) \qquad i = 1, \ldots, n \,. \tag{6.1}$$

We will consider η_0 of the form $\eta_0 = \varphi^i(u, t)\partial_i$ (where as usual $\partial_i = \partial/\partial u^i$), and proceeding as usual we get the determining equations

$$\frac{\partial F^i}{\partial\lambda} + \varphi^j \frac{\partial F^i}{\partial u^j} = \frac{\partial\varphi^i}{\partial t} + \frac{\partial\varphi^i}{\partial u^j} F^j \,. \tag{6.2}$$

Clearly, for $f = \left(f^1, \ldots, f^n\right)$ assigned, if φ_0 corresponds to a discrete symmetry and φ_* to a continuous one, then $\varphi = \varphi_0 + \varphi_*$ corresponds to a new discrete symmetry.

Therefore, in the search for discrete symmetries of a given equation Δ (equations invariant under a given discrete symmetry η_0), we should proceed modulo continuous symmetries of Δ (equations admitting η_0 as a continuous symmetry).

Let us now consider the case of time-independent symmetries, i.e. $\varphi^i = \varphi^i(u)$; in this case the determining equations

$$\frac{\partial F^i}{\partial \lambda} + \varphi^j \frac{\partial F^i}{\partial u^j} = \frac{\partial \varphi^i}{\partial u^j} F^j \tag{6.3}$$

are not only linear (in both F and φ), but also homogeneous as equations for F.

Therefore, if we consider (6.3) as an equation for F—i.e. if we look for equations invariant under a given symmetry corresponding to φ—the space of their solutions is a *module* over the algebra of functions which admits φ as a continuous symmetry. In other words, if $F_1, \ldots, F_s$ are solutions of (6.3), with $F_\alpha = (F_\alpha^1, \ldots, F_\alpha^n)$, and $\zeta_1, \ldots, \zeta_s$ are any functions invariant under η, then $F_* = \sum_{j=1}^s \zeta_j F_j$ is also a solution to (6.3).

As a concrete example of discrete symmetries for dynamical system, consider the case

$$\begin{aligned}
\dot{x} &= f_1(x, y, z) = -x + x^2 + x^2\cos(2z) - xy\sin(2z) - y^2\cos(z) \\
\dot{y} &= f_2(x, y, z) = -y + xy + xy\cos(2z) - y^2\sin(2z) + xy\cos(z) \\
\dot{z} &= f_3(x, y, z) = \cos(z) + x\cos(2z)
\end{aligned} \tag{6.4}$$

and let us look for time-independent symmetries, which we write in the form

$$\eta_0 = \varphi_1(x, y, z)\partial_x + \varphi_2(x, y, z)\partial_y + \varphi_3(x, y, z)\partial_z \, . \tag{6.5}$$

The determining equations in this case are

$$\begin{aligned}
\frac{\partial F_1}{\partial \lambda} + \varphi_1 \frac{\partial F_1}{\partial x} + \varphi_2 \frac{\partial F_1}{\partial y} + \varphi_3 \frac{\partial F_1}{\partial z} &= F_1\partial_x\varphi_1 + F_2\partial_y\varphi_1 + F_3\partial_z\varphi_1 \\
\frac{\partial F_2}{\partial \lambda} + \varphi_1 \frac{\partial F_2}{\partial x} + \varphi_2 \frac{\partial F_2}{\partial y} + \varphi_3 \frac{\partial F_2}{\partial z} &= F_1\partial_x\varphi_2 + F_2\partial_y\varphi_2 + F_3\partial_z\varphi_2 \\
\frac{\partial F_3}{\partial \lambda} + \varphi_1 \frac{\partial F_3}{\partial x} + \varphi_2 \frac{\partial F_3}{\partial y} + \varphi_3 \frac{\partial F_3}{\partial z} &= F_1\partial_x\varphi_3 + F_2\partial_y\varphi_3 + F_3\partial_z\varphi_3 \, .
\end{aligned} \tag{6.6}$$

A discrete symmetry is therefore given by

$$\eta_0 = a(y\partial_x - x\partial_y) + b\partial_z \tag{6.7}$$

with, e.g., $a = b = 1$, as shown below. Indeed, inserting this into (6.6) we get

$$\begin{aligned}
\frac{\partial F_1}{\partial \lambda} + ay\frac{\partial F_1}{\partial x} - ax\frac{\partial F_1}{\partial y} + b\frac{\partial F_1}{\partial z} &= \ aF_2 \\
\frac{\partial F_2}{\partial \lambda} + ay\frac{\partial F_2}{\partial x} - ax\frac{\partial F_2}{\partial y} + b\frac{\partial F_2}{\partial z} &= -aF_1 \\
\frac{\partial F_3}{\partial \lambda} + ay\frac{\partial F_3}{\partial x} - ax\frac{\partial F_3}{\partial y} + b\frac{\partial F_3}{\partial z} &= 0 \, .
\end{aligned} \tag{6.8}$$

If F_j are of the form

$$
\begin{aligned}
F_1 &= \zeta_1 x - \zeta_2 y \\
F_2 &= \zeta_1 y + \zeta_2 x \\
F_3 &= \zeta_3
\end{aligned}
\tag{6.9}
$$

with $\zeta_j = \zeta_j(\xi)$ arbitrary smooth functions of

$$
\xi_1 = x^2 + y^2 \qquad \xi_2 = \arctan(y/x) - a\lambda \qquad \xi_3 = z - b\lambda \tag{6.10}
$$

we get a solution to (6.8); if moreover the ζ_j are periodic of period τ_2 in ξ_2 and of period τ_3 in ξ_3, then the solutions F are invariant under $e^{\lambda_0 \eta_0}$ with λ_0 the smallest number such that $\lambda_0 = n_2\tau_2/a = n_3\tau_3/b$ with n_2, n_3 integers. It suffices to choose

$$
\begin{aligned}
\zeta_1 &= -1 + \sqrt{\xi_1}\cos(\xi_2) + \sqrt{\xi_1}\cos(\xi_2 + 2\xi_3) \\
\zeta_2 &= \sqrt{\xi_1}\sin(\xi_2)\cos(\xi_3) \\
\zeta_3 &= \cos(\xi_3) + \sqrt{\xi_1}\cos(\xi_2)\cos(2\xi_3)
\end{aligned}
\tag{6.11}
$$

in order to recover our system (6.4). Indeed, the action of $\Lambda = e^{\lambda_0 \eta_0}$ on R^3 is given by

$$
\Lambda : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x\cos(a\lambda_0) - y\sin(a\lambda_0) \\ y\cos(a\lambda_0) + x\sin(a\lambda_0) \\ z + b\lambda_0 \end{pmatrix} \tag{6.12}
$$

and the invariance of (6.4) under (6.12) can now be checked immediately, for $\lambda_0 a = \lambda_0 b = 2\pi$ (e.g. $a = b = 1$, $\lambda_0 = 2\pi$). It is also trivial to check that Λ does not reduce to the identity, and that it is not a symmetry of (6.4) for arbitrary λ_0, i.e. η_0 is not a continuous symmetry.

7. Relations with the problem of continuous symmetries of discrete equations

The problem we are discussing in this work should not be confused with another interesting problem in symmetry theory, the continuous symmetries of discrete equations. Indeed, in this case the Lie method to constructing continuous symmetries, based on the tangent space approach—i.e. on considering the infinitesimal generators of continuous transformations, which generate a Lie algebra—can be adapted for the case of discrete equations [9–13], and the Lie algebra of continuous symmetries of discrete equations can be found.

This problem is dual to the problem we consider in the present paper, in the sense that while we consider a continuous (i.e. differential) equation and discrete symmetries, in [9–13] the evolution equation is a discrete map, and one looks for continuous symmetries. Thus the roles of evolution equation and of symmetry transformation are interchanged.

This suggests that the method described in the present work could also be of interest in the case of difference or difference–differential equations, in the same way Lie method has proved to be useful for these equations.

Although we do not intend to develop this point here, we would like to stress that not only can our method be used—modulo this exchange of roles between the evolution equation and the symmetry transformation—to tackle the problem of continuous symmetries of discrete equations, but that this reformulation makes the latter problem equivalent to our 'inverse' problem. This means, in particular, that we have to solve normal PDEs, and not a functional equation.

Needless to say, this holds only when the discrete map can be written as the time-one map associated with a continuous evolution. Note that this restriction is *not* present in [9–13], so our method is definitely less generally applicable than those already available. On the other hand, as it will also become clear by the examples presented below, when applicable the method presented here requires relatively simple computations and could therefore be convenient in concrete applications of the appropriate class.

In particular, all the examples presented in the final section could be reinterpreted in this frame. To be concrete, in the case we consider a quantized symmetry with generator

$$\eta_0 = \varphi(u, x)\partial_u \tag{7.1}$$

and determine the, say, first-order ODEs

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u) \tag{7.2}$$

which admit this as quantized symmetry with, say, period one, we can reinterpret this as saying that we have a discrete evolution equation written as

$$x_{n+1} = \Phi(x) \tag{7.3}$$

and that we are looking for continuous symmetries of this of the form

$$\nu = \psi(x)\partial_x\,. \tag{7.4}$$

This is done by assuming that the discrete map is

$$\Phi = \mathrm{e}^{\lambda_0\eta_0} \tag{7.5}$$

for a convenient λ_0 (e.g. $\lambda_0 = 1$); now the ν will be given by (7.4) with

$$\psi(x) = f(x) \tag{7.6}$$

where the f are those determined above, i.e. the same as in (7.2).

Similar consideration would apply for more complicate situations. For example, if we discretize a continuous equation and construct a difference–differential equation Φ (for instance in a lattice), and consider its continuous symmetries η_0, this setting can be reinterpreted saying that we have a continuous evolution—corresponding to η_0—with discrete and continuous symmetries corresponding to Φ.

A detailed analysis of the possible applications of our method to the problem of symmetries of discrete and/or difference–differential equations would certainly be of interest, but lies outside the scope of the present paper. We hope to be able to report on this in the near future.

8. Geometrical interpretation

Let us first go back to considering quantized symmetries for manifolds, see section 2.

When introducing the function $F(\lambda, x)$ we are implicitly passing to the consideration of a vector field η_0' associated with η_0 and acting in $M' = R \times M$, given explicitly by

$$\eta_0' = \eta_0 + \partial_\lambda \equiv \varphi^i \partial_i + \psi \partial_y + \partial_\lambda \,. \tag{8.1}$$

Then the graph $\Theta = \{(x, y, \lambda) : y = F(\lambda, x)\} \subset M'$ of $F(\lambda, x)$ is by construction an invariant submanifold of M' under η_0'. We are therefore brought back to the problem of determining tangent vector field to a manifold; however, the manifold is now not given *a priori*, but depends itself on the vector field.

We would like to stress that M' can be naturally seen as the total space of a fibre bundle [16, 17] B with base R (corresponding to the λ), fibre M and projection $\pi : (\lambda, x, y) \to \lambda$; η_0' is then a connection on B. When we are looking for solutions to (2.6), (2.7) which moreover satisfy the periodicity condition (2.8)—i.e. we are looking for quantized symmetries—we can consider the analogous bundle $\mathcal{B}$ with S^1 as the base space, M as the fibre and the same projection π.

Our problem can then be described in differential geometric language as the search for a connection [16, 17] ∇ on $\mathcal{B}$ such that there is a section Θ invariant under this connection and such that the restriction of Θ to $\pi^{-1}(0)$ is the prescribed Γ.

Remark 12. In the above language, the determination of manifolds invariant under a given quantized symmetry (the 'inverse' problem) amounts to the determination of sections of $\mathcal{B}$ invariant under a given connection ∇. Again, it is obvious that this is much easier than the 'direct' problem, although in general it is not trivial at all.

Remark 13. It should be stressed that although Θ is an invariant manifold under the connection ∇, this does not imply that transporting a point $(x, f(x))$ around the base space S^1 by ∇ we get the same point. In general, we get a point $(x', f(x'))$ with $x' \neq x$, and the discrete transformation $\Lambda : \Gamma \to \Gamma$ is related to the holonomy [16,17] of the connection ∇.

In the case of differential equations of order n, the setting is quite similar, provided we consider different geometrical objects.

Indeed, now we should consider the nth prolongation of η_0', which we denote by η', acting in $M'^{(n)}$. But from (8.1) and the prolongation formula we can equally well consider, instead than η', the vector field

$$\Psi = \partial_\lambda + \eta \tag{8.2}$$

(with η the nth prolongation of η_0) and correspondingly, instead than $M'^{(n)}$, consider

$$P = R \times M^{(n)} \tag{8.3}$$

(where the R factor corresponds to λ). Indeed, not only does

$$\Psi : P \to TP \tag{8.4}$$

but it is also clear that the non-trivial part of the action of η' on $M'^{(n)}$ is fully embodied in the action of Ψ on P.

We can now consider a fibre bundle D with total space P, base space R and natural projection $\pi : P \to R$; the fibres of this are $\pi^{-1}(0) = M^{(n)}$.

If we are looking for quantized symmetries, we should consider the analogous bundle $\mathcal{D}$ with S^1 as base space, $M^{(n)}$ as fibre and the same projection π.

When looking for quantized symmetries of the differential equation Δ, we have then to look for a connection ∇ on $\mathcal{D}$ such that there is a section Ξ invariant under this connection and such that $\Xi(0) = D$.

However, this connection or more precisely its vertical part—differently from what happens when we deal with ordinary manifolds—must preserve the contact structure in $M^{(n)}$, as it is indeed ensured by the fact that in (8.2) we have η, i.e. the prolongation of a Lie-point vector field, and not a generic vector field in $M^{(n)}$.

Again, although Ξ is invariant under ∇, and $e^{\lambda_0 \eta} : \Delta \to \Delta$, the discrete map $\Lambda = e^{\lambda_0 \eta}|_\Delta$ does not need to be the identity (in which case we get a trivial symmetry), and it is indeed related to the holonomy of the connection ∇.

9. Examples

Here we discuss in detail some special cases, i.e. special kinds of equations, and examples. They can be treated very uniformly (at least for autonomous equations). We would like to stress some interesting points arising from the following discussion. First, it is possible to find the general form of the equation, depending on an arbitrary periodic function. Second, the (finite) transformation can be in some sense linearized, transforming the generic one-parameter group into an additive one (translations).

9.1. Autonomous first-order ODE

Let us consider the simplest example, an *autonomous first-order* ODE, $u_t = f(u)$, and try to find the function f in order that it admits a discrete symmetry, with associated vector field (for the corresponding continuous symmetry), $\eta_0 = \varphi(u)\partial u$. Take an arbitrary vector field for the autonomous equation, $\eta_0 = \varphi(u)\partial_u$ and write the determining equation (3.4)

$$\frac{\partial F}{\partial \lambda} + \varphi(u)\frac{\partial F}{\partial u} = \varphi_u F \,. \tag{9.1}$$

This linear equation can be solved using characteristics. Though the equation is rather undetermined (we do not know $\varphi(u)$ nor $F(u, \lambda)$), we can find a general form for $f(u)$, in terms of $\varphi(u)$ and some arbitrary periodic function. The characteristic equations are

$$d\lambda = \frac{du}{\varphi} = \frac{dF}{\varphi_u F} \tag{9.2}$$

where φ depends only on u and F on u and λ. This allows us to solve the first equation, with the solution

$$C = \int \frac{du}{\varphi(u)} - \lambda \tag{9.3}$$

where C is a constant. Using the second equation in (9.2), we get

$$\int \frac{\varphi_u(u)du}{\varphi(u)} = \int \frac{dF}{F} \tag{9.4}$$

giving the solution

$$F(u, \lambda) = h\left(\int \frac{\mathrm{d}u}{\varphi(u)} - \lambda\right)\varphi(u) \tag{9.5}$$

where h is a periodic function. Then, the function f in the ODE is simply

$$f(u) = h\left(\int \frac{\mathrm{d}u}{\varphi(u)}\right)\varphi(u) \tag{9.6}$$

and the differential equation having a discrete symmetry (associated with a continuous one given by η) is

$$u_t = h\left(\int \frac{\mathrm{d}u}{\varphi(u)}\right)\varphi(u)\,. \tag{9.7}$$

As a particular case, if we take h to be constant, we have the continuous symmetry for the differential equation $u_t = \varphi(u)$

If we specialize the vector field η_0 and the periodic function $h(u)$ we can find, for instance, invariant equations under translations, scale and special conformal transformations. In fact, take $h(u) = \sin u$ and $\varphi(u) = 1$. The equation is $u_t = \sin u$. For scale transformations, $\varphi(u) = u$, we get $u_t = u \sin(\log u)$. And finally, for special conformal transformations, $\varphi(u) = u^2$, we have $u_t = u^2 \sin(1/u)$.

We remark that the argument of h is the function which is added to the parameter λ in the finite transformation: thus, in translations we have $u \to u + \lambda$; in scale transformations we have $u \to \mathrm{e}^{\lambda} u$ and therefore $\log u \to \log u + \lambda$; and in special conformal transformations we have $u \to u/(1 + \lambda u)$ and therefore $(1/u) \to [(1/u) + \lambda]$.

This remark gives us another point from which to view this example. Take the autonomous equation

$$u_t = f(u) \tag{9.8}$$

and consider a transformation in the u variable, with parameter λ:

$$u' = g(u, \lambda)\,. \tag{9.9}$$

Now, let us find a change of variable $z = H(u)$ linearizing the above transformation, that is, the transformed z' is

$$z' = z + \lambda \tag{9.10}$$

or, in terms of the old variable u

$$H(u') = H(u) + \lambda \qquad g(u, \lambda) = H^{-1}(H(u) + \lambda)\,. \tag{9.11}$$

In terms of the new variable, z, the differential equation is

$$z_t = H'(H^{-1}(z))\, f(H^{-1}(z)) = \tilde{f}(z) \tag{9.12}$$

where H' is the derivative of H with respect to its argument. If this equation is invariant under a discrete translation (for instance with $\lambda = 2\pi$), $z' = z + 2\pi$, the function $\tilde{f}$ should be a periodic function. The original f is given by

$$f(u) = \tilde{f}(H(u))\frac{1}{H'(u)} \tag{9.13}$$

which is the form (9.6) we obtained through the determining equation, because $\tilde{f}$ is a periodic function in the variable which is used to turn the transformation into a translation (in the variable z), and $1/H'(u)$ is the corresponding vector field. This is easily computed. The infinitesimal transformation associated with the translations (9.10) corresponds (after the change of variable given by (9.11)) to

$$\varphi(u) = \left.\frac{\partial}{\partial\lambda}H^{-1}(H(u)+\lambda)\right|_{\lambda=0} = \frac{1}{H'(u)} \tag{9.14}$$

which is the result we got from the determining equation (9.6), (9.13).

9.2. Autonomous second-order ODE

This case is close to the previous one, though the equations are slightly more complicated. Let us consider the equation

$$u_{tt} = f(u, u_t) \tag{9.15}$$

and the vector field $\eta_0 = \varphi(u)\partial u$.

The determining equation (3.10) is

$$\frac{\partial F}{\partial\lambda} + \varphi\frac{\partial F}{\partial u} + \varphi_u u_t\frac{\partial F}{\partial u_t} = \varphi_{uu}u_t^2 + \varphi_u F \tag{9.16}$$

which is again a linear equation and can be solved by characteristics

$$\mathrm{d}\lambda = \frac{\mathrm{d}u}{\varphi} = \frac{\mathrm{d}u_t}{\varphi_u u_t} = \frac{\mathrm{d}F}{\varphi_u F + \varphi_{uu}u_t^2}\,. \tag{9.17}$$

The first equation is solved in the same way we did above for (9.2), and its solution is given by (9.3), i.e. $C_1 = \int (du/\varphi) - \lambda$. From the second equation we get

$$\int\frac{\varphi_u(u)\,\mathrm{d}u}{\varphi(u)} = \int\frac{\mathrm{d}u_t}{u_t} \tag{9.18}$$

with solution

$$C_2 = \frac{u_t}{\varphi}\,. \tag{9.19}$$

Finally, the last equation is

$$\frac{\mathrm{d}u}{\varphi} = \frac{\mathrm{d}F}{\varphi_u F + \varphi_{uu}u_t^2}\,. \tag{9.20}$$

This is a inhomogeneous linear equation and its solution is

$$F(u, u_t, \lambda) = h\left(\int\frac{\mathrm{d}u}{\varphi(u)} - \lambda, \frac{u_t}{\varphi(u)}\right)\varphi(u) + \frac{u_t^2}{\varphi(u)}\varphi_u(u) \tag{9.21}$$

where h is an arbitrary function satisfying the periodicity condition on λ. If the dependency on u_t is eliminated, we get the same result as in the first example (9.5).

The function f appearing in the equation is then

$$f(u, u_t) = h\left(\int\frac{\mathrm{d}u}{\varphi(u)}, \frac{u_t}{\varphi(u)}\right)\varphi(u) + \frac{u_t^2}{\varphi(u)}\varphi_u(u)\,. \tag{9.22}$$

9.3. Autonomous first-order PDE

It turns out that the discussion of 'autonomous first-order PDE' equations is just the same as that for ODE's. Consider the equation

$$u_t = f(u, u_x) \tag{9.23}$$

where x is a vector variable, $x = (x_i)$, and write the determining equation (3.20)

$$\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + \varphi_u u_i \frac{\partial F}{\partial u_i} = \varphi_u F \tag{9.24}$$

and again, φ is a function of u, and $u_i = \partial_{x_i} u$. This is a linear equation and can be solved using characteristics as in the previous examples. The characteristic equations are

$$\mathrm{d}\lambda = \frac{\mathrm{d}u}{\varphi} = \frac{\mathrm{d}u_i}{\varphi_u u_i} = \frac{\mathrm{d}F}{\varphi_u F}\,. \tag{9.25}$$

As φ is a function of u, we can solve the first equation, which is again (9.2), with solution (9.3) $C_0 = \int (\mathrm{d}u/\varphi(u)) - \lambda$ and for $i = 1, \ldots, n$ the equations (9.18), $\mathrm{d}u/\varphi = \mathrm{d}u_i/\varphi_u u_i$ with solutions (9.19), $C_i = u_i/\varphi,\ i = 1, \ldots, n$.

Using the equation (9.4) we obtain the general solution

$$F(u, u_i, \lambda) = h\left(\int \frac{\mathrm{d}u}{\varphi(u)} - \lambda, \frac{u_1}{\varphi}, \ldots, \frac{u_n}{\varphi}\right)\varphi(u) \tag{9.26}$$

where h is a function periodic in λ. The function $f(u, u_1, \ldots, u_n)$ is just

$$f(u, u_1, \ldots, u_n) = h\left(\int \frac{\mathrm{d}u}{\varphi(u)}, \frac{u_1}{\varphi}, \ldots, \frac{u_n}{\varphi}\right)\varphi(u)\,. \tag{9.27}$$

As a particular example, the function h could be chosen as

$$h(\lambda, u, u_1, \ldots, u_n) = h_0\left(\int \frac{\mathrm{d}u}{\varphi(u)} - \lambda\right) g\left(\frac{u_1}{\varphi}, \ldots, \frac{u_n}{\varphi}\right) \tag{9.28}$$

with h_0 a periodic function. In this way the solution is similar to that we got in the ODE case (9.5), with a new factor, given by the function g, depending on the derivatives of u with respect to the variables x_i.

9.4. Autonomous second-order PDE

Though the determining equation for

$$u_{tt} = f(u, u_t, u_x, u_{xt}, u_{xx}) \tag{9.29}$$

is rather cumbersome in this case, its solution is the same as that found in the previous example (modulo a change of the corresponding invariants), i.e.

$$\begin{aligned}\frac{\partial F}{\partial \lambda} + \varphi \frac{\partial F}{\partial u} + \varphi_u \left(u_x \frac{\partial F}{\partial u_x} + u_t \frac{\partial F}{\partial u_t}\right) + \left(\varphi_{uu} u_x^2 + \varphi_u u_{xx}\right) \frac{\partial F}{\partial u_{xx}} + \left(\varphi_{uu} u_x u_t + \varphi_u u_{xt}\right) \frac{\partial F}{\partial u_{xt}} \\ = \varphi_{uu} u_t^2 + \varphi_u F\end{aligned} \tag{9.30}$$

so we will not give the details here. The solution is, when $\varphi = \varphi(u)$,

$$F(\lambda, u, u_t, u_x, u_{xx}, u_{xt}) = h\left(\int \frac{\mathrm{d}u}{\varphi(u)} - \lambda, \frac{u_x}{\varphi}, \frac{u_t}{\varphi}, \frac{\varphi u_{xx} - u_x^2\varphi_u}{\varphi^2}, \frac{\varphi u_{xt} - u_x u_t \varphi_u}{\varphi^2}\right)\varphi(u) + \frac{u_t^2}{\varphi}\varphi_u \tag{9.31}$$

where h is a periodic function in λ, and f is obtained setting $\lambda = 0$.

The sine-Gordon equation corresponds to a particular case of this function. Taking $\varphi(u) = 1$, so that

$$F(\lambda, u, u_t, u_x, u_{xx}, u_{xt}) = h\,(u - \lambda, u_x, u_t, u_{xx}, u_{xt}) \tag{9.32}$$

and suitably choosing the function h we get

$$F(\lambda, u, u_{xx}) = u_{xx} + \sin(u - \lambda)\,. \tag{9.33}$$

Then F is a periodic function in λ and its value in $\lambda = 0$ is

$$f(u, u_{xx}) = u_{xx} + \sin u\,. \tag{9.34}$$

9.5. *Non-autonomous first-order* ODE

As we have seen in the previous examples, the case of autonomous equations and vector fields depending only on the variable u can be solved completely, giving a general form for the equations admitting this type of symmetries. Let us now study the case of a non-autonomous equation

$$u_t = f(t, u) \tag{9.35}$$

with vector fields of the type

$$\eta_0 = \tau(t, u)\partial_t + \varphi(t, u)\partial_u\,. \tag{9.36}$$

The determining equation is (3.7) which is no longer a linear equation, though the nonlinearity is localized in the term F^2. The main problem is, however, the dependency of the functions τ and φ on t, u. We cannot obtain a general solution in terms of these functions as we did in the autonomous case. Here, we should specify the explicit expression of the vector field in order to obtain some interesting results.

Let us consider the following vector field:

$$\eta_0 = au\partial_u + bt\partial_t \tag{9.37}$$

where a and b are two fixed non-zero constants (the discussion to follow can be easily adapted to the case where one of them is zero). In this case, the determining equation becomes linear

$$\frac{\partial F}{\partial \lambda} + bt\frac{\partial F}{\partial t} + au\frac{\partial F}{\partial u} = (a - b)F \tag{9.38}$$

and its solution can be computed easily. We can find the two invariants

$$C_1 = t\mathrm{e}^{-b\lambda} \qquad C_2 = t^a u^{-b} \tag{9.39}$$

and the solution

$$F(\lambda, t, u) = h(t\mathrm{e}^{-b\lambda}, t^a u^{-b})u^{1-b/a} \tag{9.40}$$

which should be periodic in λ. We can take, as a particular example,

$$F(\lambda, t, u) = g(t^a u^b)u^{1-b/a} \sin(\log(u\mathrm{e}^{-a\lambda})) \tag{9.41}$$

and the equation

$$u_t = g(t^a u^{-b})u^{1-b/a} \sin(\log u) \tag{9.42}$$

where g is an arbitrary function. We remark that these equations can be written in many different ways, due to the different choices of the invariants coming from the characteristic method. The arbitrariness of the function g comes from its invariance under the continuous symmetry. The functions sin and log come from the discrete symmetry ($\lambda \to \lambda + 2\pi/a$, where a can be taken equal to one).

An example of nonlinear determining equations is given by the two following vector fields (Lorentz transformations and rotations). Let us first consider the vector field

$$\eta_0 = t\partial_u + u\partial_t \tag{9.43}$$

and the quasilinear equation

$$\frac{\partial F}{\partial \lambda} + u\frac{\partial F}{\partial t} + t\frac{\partial F}{\partial u} = 1 - F^2 . \tag{9.44}$$

We can compute two invariants

$$C_1 = \lambda - \log(u + t) \qquad C_2 = t^2 - u^2 \tag{9.45}$$

and the solution

$$F(\lambda, t, u) = \frac{u + h(C_1, C_2)t}{t + h(C_1, C_2)u} . \tag{9.46}$$

We should impose periodicity conditions in λ on the function h, and then take $\lambda = 0$ to get the differential equation.

Finally, using infinitesimal rotations

$$\eta_0 = t\partial_u - u\partial_t \tag{9.47}$$

the determining equation is

$$\frac{\partial F}{\partial \lambda} - u\frac{\partial F}{\partial t} + t\frac{\partial F}{\partial u} = 1 + F^2 . \tag{9.48}$$

The two invariants are

$$C_1 = \lambda - \arcsin \frac{u}{\sqrt{t^2+u^2}} \qquad C_2 = t^2 + u^2 \tag{9.49}$$

and the solution is

$$F(\lambda, t, u) = \frac{u + h(C_1, C_2)t}{t - h(C_1, C_2)u}\,. \tag{9.50}$$

If we choose

$$h(C_1, C_2) = \sin\left(k \arcsin \frac{u}{\sqrt{t^2+u^2}} - k\lambda\right) \tag{9.51}$$

where k is a non-zero constant, we get as a differential equation with a discrete symmetry

$$u \to u\cos(2\pi/k) - t\sin(2\pi/k) \qquad u \to u\sin(2\pi/k) + t\cos(2\pi/k) \tag{9.52}$$

$$u_t = \frac{u + t\sin\left(k \arcsin \frac{u}{\sqrt{t^2+u^2}}\right)}{t - u\sin\left(k \arcsin \frac{u}{\sqrt{t^2+u^2}}\right)}\,. \tag{9.53}$$

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